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Regular Quantum Field Theories

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Abstract

Mathematically regular and more precise versions of quantum field theories are discussed. A new class of representations called "minimal wave-packet representations" is introduced. Several possibilities of constructing nonconventional, bounded interaction operators (nonpolynomial, nonlinear, explicitly or implicitly nonlocal) corresponding to the traditional φ^4 or φ^3 interactions are reviewed. The problem of macrocausality is discussed. A procedure of renormalization of regular theories is indicated.

1. *Minimal Wave-Packet Representations (MWPR)*

Quantum field theory, being a theory of systems with infinitely many degrees of freedom, presents serious difficulties and peculiarities. One of the peculiarities was exhibited by the discovery of unitarily inequivalent representations. It compels one to point out a particular representation (among the variety of representations seemingly standing on equal footing) as physically admissible and regard every other representation unitarily equivalent to this particular one as equally well acceptable, and every inequivalent as inadmissible.

Any particular representation is connected with eigenvectors of a complete set of observables which, in turn, are related in quantum field theory to a spacelike hypersurface or a hyperplane t = const. of a particular inertial frame of reference. Therefore a choice of a particular representation distinguishes a particular coordinate system while the Lorentz covariance has to be demonstrated by showing that two representations connected with two different Lorentz frames are unitarily equivalent (belong to the same equivalence class).

Let us introduce an inertial-Cartesian coordinate system x^μ and consider the field quantities $\psi(\mathbf{r})$ at a fixed time instant, say $t = 0$. Consider also a six-dimensional phase space of the coordinates $\mathbf{r}$ and of the arguments $\mathbf{k}$ of the

Fourier-transformed field quantities

$$\psi(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3x e^{-i\mathbf{kr}} \psi(\mathbf{r}) \tag{1.1}$$

Let us divide this phase space into cells of magnitude h^3 (unit cells in natural units $h = c = 1$) and enumerate them by an index n in an arbitrary but unique way. With each cell let us connect a function

$$\psi_n(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3k \, e^{i(\mathbf{k} - \mathbf{k}_n)(\mathbf{r} - \mathbf{r}_n)} e^{-\lambda^2(\mathbf{k} - \mathbf{k}_n)^2} \tag{1.2}$$

and its Fourier transform

$$\psi_n(\mathbf{k}) = \frac{1}{2(\pi)^{3/2}} \int d^3x \, e^{-i(\mathbf{k} - \mathbf{k}_n)(\mathbf{r} - \mathbf{r}_n)} e^{-(\mathbf{r} - \mathbf{r}_n)^2/\lambda^2} \tag{1.3}$$

where the components of $\mathbf{k}_n$ and $\mathbf{r}_n$ denote a point in the nth cell of the phase space. As is well known, such Gaussian distributions yield a minimum of the dispersion products $\Delta\mathbf{r} \cdot \Delta\mathbf{k}$.

Consider, e.g., a complex scalar field ψ describable classically within the traditional framework of a Lagrangian and Hamiltonian formalism and assume that the most general form of a physically acceptable field at any time instant is representable as a superposition of the functions (1.2)

$$\psi(\mathbf{r}) = \sum_n q^{(n)} \psi_n(\mathbf{r}), \qquad \pi(\mathbf{r}) = \sum_n p^{(n)} \psi_n(\mathbf{r}) \tag{1.4}$$

where π is the momentum canonically conjugate to the field quantity ψ. This assumption settles the meaning of completeness in the functional space $\{\psi\}$ (Bargmann, et al., 1971; Perelomov, 1971).

Field quantization is to be performed by regarding the enumerable set of coefficients $q^{(n)}$ and $p^{(n)}$ as operators chosen so as to satisfy the usual canonical (equal time) commutation relations

$$[\pi(\mathbf{r}), \psi(\mathbf{r}')] = \frac{1}{i} \delta^{(3)}_{(\mathbf{r} - \mathbf{r}')}, \qquad [\psi(\mathbf{r}), \psi(\mathbf{r}')] = [\pi(\mathbf{r}), \pi(\mathbf{r}')] = 0 \tag{1.5}$$

Novel in this procedure is the circumstance that the functions ψ_n do not constitute an orthogonal set:

$$\int d^3x \psi^*_m(\mathbf{r}) \psi_n(\mathbf{r}) = g_{mn} \tag{1.6}$$

with

$$g_{nn} = 1 \text{ but } g_{nm} \neq 0 \text{ for } m \neq n \tag{1.6'}$$

Consequently, the commutation relations between the coefficients of the decomposition (1.4) are slightly more complicated than usually:

$$[p^{(n)}, q^{(m)}] = (1/i) g^{nm} \tag{1.7}$$

where g^{nm} are defined by the relations

$$g^{nk}g_{km} = \delta_m{}^n \tag{1.8}$$

The relations (1.7) may be satisfied with the help of the usual creation and annihilation operators introduced by the following relations:

$$q^{(n)} = \frac{1}{(2\omega_n)^{1/2}}\,[a(n) + b^\dagger(n)]\,, \qquad p^{(n)} = \frac{1}{i}\,g^{nm}\left(\frac{\omega m}{2}\right)^{1/2}\,[b(m) - a^\dagger(m)] \tag{1.9}$$

using the summation convention for the index m in (1.9). At first, ω_n is an arbitrary coefficient.

It should be noticed that in the case of the traditional formulations in terms of creation and annihilation operators $a^\dagger(\mathbf{k})$ and $a(\mathbf{k})$ for particles with exactly fixed momenta $\mathbf{k}$, one was compelled, later on, to introduce an auxiliary concept of the so called "test functions." In our case this is not necessary because the functions (1.2) serving for construction of the basis of the minimal wave packets' representation (MWPR) are themselves physically satisfactory test functions.

The physical meaning of the creation and annihilation operators $a^\dagger(n)$, $a(n)$, $b^\dagger(n)$, $b(n)$ introduced above is obvious: They apply to particles characterized dynamically by the fact of belonging to the nth cell of the phase space, i.e., their positions and momenta are known up to the usual quantum mechanical uncertainties. Thus, the representations described above are those of the numbers of particles represented by minimal wave packets, i.e., with products of momentum and position inaccuracies reaching a minimum (with the exclusion of the unphysical extreme cases of infinitely sharp localization at the cost of a complete lack of information about momentum or vice versa). Therefore such representations (MWPR) deserve to be also called "proper" or "fundamental."

Inasmuch as the number of cells in the phase space is enumerable, these representations possess enumerable bases (the $\mathscr{H}$ space is separable) similarly as in the case of momentum representations in a periodicity box. Each vector of the basis as well as their finite superpositions represent well-localized states, i.e., all particles are practically restricted to finite distances from the origin of the coordinate system while, on the other hand, the position of neither of them is infinitely sharp. The space of physically admissible state vectors is the Schwarz space (S space).

Every representation of this type introduces a kind of crystal structure with nodes (cites) at the centers of the minimal phase-space cells. This reminds one of Wilson's concept of a discrete space (Wilson, 1974), but in our case such structures are purely abstract and the theory remains translationally invariant. Indeed, two representations differing from one another by a shift of the cells are unitarily equivalent. Also, a transition between two such representations differing from one another by a Lorentz transformation is effected by a unitary transformation. Besides the Lorentz transformations and translations

of the cells also their shape may be changed, e.g., by changing the value of the constant λ appearing in (1.2) and (1.3) provided we keep $0 < \lambda < \infty$.

It should also be pointed out that the departure from orthogonality of the set of functions (1.2) spanning the representation is very small:

$$g_{mn} \simeq \delta_{mn} \tag{1.10}$$

In the approximation consisting in a replacement of g_{mn} by δ_{mn}, and taking λ very large, the MWPR exhibit all the advantages of momentum representations in a finite box. With such representations neither an artificial concept of a periodicity box nor a clumsy procedure of introducing a cutoff in the x space is necessary.

To conclude this section let us postulate that every representation unitarily equivalent to the MWPR described above is physically admissible and proper, whereas every other, inequivalent, representation is to be rejected.

2. *Bounded Interactions*

Quantum field theories formulated in the representations described above and with interactions given by bounded operators will be called "regular field theories." There exist, a priori, several possibilities of constructing bounded interaction operators, and it seems worthwhile to discuss here briefly some of them.

2.1 *Nonpolynomial Interactions.* The most suggestive possibility of constructing a bounded interaction in field theory is to take, as an interaction Lagrangian density $\mathscr{L}'$, a bounded function of the field quantities, e.g., the famous Sine-Gordon interaction

$$\mathscr{L}' = l^{-4} g(\cos l\varphi - 1) \tag{2.1}$$

where φ is a real scalar field, g is a dimensionless coupling constant, and l is a constant with dimension of length (in natural units). The first term of the power series expansion of (2.1) contributes to the bare mass, whereas the next term yields the famous φ^4 interaction. The trouble with such form of interaction is that, originally, the interaction term appears without Wick's normal ordering, whereas if performing the ordering of the Lagrangian $\mathscr{L}' \to :\mathscr{L}':$ the separate terms of the expansion become meaningful but the boundedness of the Lagrangian as a whole is destroyed. Indeed, putting $\cos l\varphi$ in between the colons means subtracting infinite terms.

Several authors expressed the opinion that in every case of nonpolynomial Lagrangians the effective interaction disappears, i.e., the field is effectively a free field. We are not quite convinced by their arguments and expect that, at least in some cases, a bounded nonpolynomial Lagrangian density may yield effectively a nontrivial theory. In particular, a bounded function of a normally ordered operator (but not a function ordered as a whole), e.g.,

$$\mathscr{L}' = g l^{-4} \arctan l^4 :\varphi^4:(x) \tag{2.2}$$

seems to be well defined, bounded, and not effectively equal to zero. But we admit that a rigorous proof of this supposition is unknown yet and the problem needs further investigation.

2.2. *A Nonlinear Interaction.* Interaction operators do not need to be necessarily defined analytically as polynomials or infinite series of the field quantities but may be as well introduced by defining how they act upon all bra-ket vectors. Let us consider the following example of a bounded interaction corresponding to the φ^4 interaction (or any other power of φ) in the weak field limit. To this end let us point out that with each linear operator 0 there is associated a function $\langle\Phi|0|\Psi\rangle$ attaching a number to each pair of bra $\langle\Phi|$ and ket $|\Psi\rangle$ but not vice versa: It is always possible to define functions attaching a number to each pair of vectors $\langle\Phi$ and $\Psi\rangle$ which are not connected with linear operators. Let us denote such functions by the following symbol:

$$F(\Phi, \Psi) = \langle\Phi * F * \Psi\rangle \tag{2.3}$$

where vertical lines (being symbols of disjunction or separability) have been replaced by asterisks (being symbols of conjunction). Indeed, we are not allowed to detach $\langle\Phi$ or $\Psi\rangle$ from (2.3) to get a ket or bra vector. It is also not allowed to replace an asterisk by a unit operator

$$* \to \mathbb{1} = \sum_n |A_n\rangle\langle A_n| \tag{2.4}$$

if F is not a linear function of both $\langle\Phi$ and $\Psi\rangle$.

After these preliminaries let us define the following interaction function

$$\langle\Phi * \mathscr{H} * \Psi\rangle = \begin{cases} g\langle\Phi|:\varphi^4:|\Psi\rangle & \text{if } |\langle\Phi|:\varphi^4:|\Psi\rangle|^2 \leqslant l^{-8} \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

with l denoting a fundamental length.

At first sight one is inclined to reject decisively such interactions because every child knows that a probabilistic interpretation is possible only if observables are represented by linear operators. The point is, however, whether the energy density (Hamiltonian) is an observable in the strict sense or not. It is already at the classical level that the Hamiltonian is sharply distinguished among functions of generalized coordinates and momenta inasmuch as it is that particular function that cannot, itself, play the role of a generalized momentum because a coordinate canonically conjugate to it is not well defined. We may assume that (besides spin and unitary spin variables) only generalized coordinates and momenta are observables in the strict sense, whereas the Hamiltonian density is not. Consequently, the energy representation does not belong to the physically acceptable representations, in agreement with the results of Section 1.

In the case of a nonlinear Hamiltonian of interaction the dynamics may be formulated on the following axiomatic basis: If the initial vector is $|t_0\rangle$ and we are interested in measuring (a complete set of) observables denoted for

brevity A at the instant t, then we have to solve the Schrödinger equation in this particular representation

$$\sum_m \int d^3x \langle A_n * \mathscr{H} * A_m \rangle \langle A_m | t \rangle = i\hbar \langle A_n | t \rangle \tag{2.6}$$

with initial $\langle A_n | t_0 \rangle$ given. The quantity $\langle A_n | t \rangle$ computed from (2.6) is just the probability amplitude in question. If, on the other hand, we were interested in a measurement, at the instant t, of another (complete) set of observables B, then we should solve an inequivalent to the former equation in the B representation but not project the solution of (2.6) upon the basis vectors $|B_m\rangle$. In this way it is seen that within the framework of this formalism there exist many inequivalent Schrödinger equations yielding unitarily inequivalent results: a "fan" of vectors $|t\rangle$ resulting from one and the same initial vector $|t_0\rangle$. This is acceptable (the more so as the ordinary quantum field theory suffers from the appearance of inequivalent representations) but shows clearly that it does not make sense to speak about the concept of "state" as such (describing immanently the system) but only about some recipes or prescriptions of how to evaluate probabilities of outcomes of prospective measurements.

One should keep open eyes for the possibility of introducing bounded and essentially nonlinear interactions, one example of them having been described above.

2.3. *Implicitly Nonlocal Interactions.* It seems to be possible to deal with renormalizable as well as with nonrenormalizable theories in the following way: First of all, use the (well known to mathematicians) prescriptions of regularizing the results of the theory by taking the finite part of Hadamard (FP) from any expression (integral) appearing in the course of computations (Gelfand and Shilov, 1958). Then perform a renormalization of the constants appearing explicitly in the Lagrangian and Hamiltonian (masses, coupling constants). Moreover, perform a renormalization of the vacuum energy (by a suitable subtraction) and—if necessary—perform also a renormalization of the probability amplitudes. The finite results obtained in this way may be assumed to describe just the physical reality, notwithstanding whether the theory with which we started was renormalizable or not renormalizable.

However, if the theory is nonrenormalizable, i.e., if the dimension of the coupling constant is a positive power of length,

$$g = l^x \quad \text{with} \quad x > 0 \tag{2.7}$$

the resulting theory, truncated by taking FP, becomes nonlocal with the nonlocality range of the order of magnitude of l. Such theory deserves the name of implicitly nonlocal in contradistinction to a theory discussed in the next section.

It should be noticed that the procedure of extracting a finite part FP differs from most of the regularization procedures (cutoffs) because in this case the cutoff is not to be removed at the end of the computations (after renormalization).

In contradistinction to other alternatives of regular theories discussed in this paper it is not obvious whether the whole perturbative series converges to a finite result, or whether it is only an asymptotic series.

2.4. *Explicitly Nonlocal Interactions.* Historically the earliest attempts at formulating a field theory with bounded interaction were the explicitly nonlocal theories involving a form factor in the Lagrangian of interaction (Peierls, and McManus, 1948; Rayski 1951, 1963). The nonlocal analog of a φ^3 interaction is derivable from an action principle (Hamilton principle) with the following action integral:

$$W'_{21} = g \int_{\sigma_1}^{\sigma_2} dx \int dr' \int dr'' \int dr''' F(r', r'', r''') : \varphi' \varphi'' \varphi''' : \tag{2.8}$$

where, besides the four-point x, there are involved further three four-points x', x'', x'''. For abbreviation φ' denotes $\varphi(x')$, etc., whereas r' etc., denote the relative coordinates

$$r' = x' - x, \qquad r'' = x'' - x, \qquad r''' = x''' - x \tag{2.9}$$

It should be noticed that only one integral in (2.8) is extended between the finite limits (between two spacelike hypersurfaces of intended measurements), whereas the remaining integrations extend over the whole space-time. This is necessary to secure the additivity of the action integrals

$$W_{31} = W_{32} + W_{21} \tag{2.10}$$

It is seen that the limits of integrations (the hypersurfaces of intended initial and final measurements σ_1 and σ_2) are built into the manifestly nonlocal theory more intrinsically than was the case with the local theories, because now the Lagrange equations are mixed integrodifferential equations with one of the integrations extended between these limits so that also their solutions must be functionals of these limits:

$$\varphi = \varphi(x, \sigma_1, \sigma_2) \tag{2.11}$$

Consequently, these hypersurfaces are not merely auxiliary, abstract concepts, but must possess an objective meaning: They must be *the* hypersurfaces where an initial and a final state have to be determined by means of *actual* measurements.

It was pointed out repeatedly and with great stress by Niels Bohr that in order to speak wisely about physical phenomena one should not treat the physical system quite abstractly but relate it to the measuring apparatus and to the well-defined conditions of measurement. Also the evolution of physical systems in time may depend objectively upon when and where the measuring apparatus was used. In a nonlocal theory it is not strange and unexpected that the hypersurfaces of measurements may interfere with the time evolution of the system between them, i.e., that the solutions of the field equations may slightly depend (functionally) on them. This is an unavoidable sign of violation of microcausality.

With a suitably chosen form factor the Lagrangian density of interaction is bounded and the Lagrangian equations are regular, i.e., their solutions are analytic functions of the coupling constant. Therefore it is legitimate to start with quantizing the free fields in a MWPR described in Section 1 (or in a momentum representation in a periodicity box) and to derive the operators representing the interacting fields from those describing free fields by means of the usual perturbation calculus.

In spite of an essential progress in the problem of securing convergence, the manifestly nonlocal theories were rejected by a great majority of experts mainly because of the following reason: Relativistic form factors of the type appearing in (2.8) seem to be incompatible with the requirement of macrocausality. Being Lorentz invariant they are functions of the squared four-vectors r^μ and their scalar products and assume the same (finite) values whenever these scalars are small constants, which occurs not only when all components of these vectors are small separately but also along the light-cone where their timelike and spacelike components separately may be arbitrarily large.

In order to secure macrocausality (Peierls and McManus, 1948; Chretien and Peierls, 1953; Arnous and Heitler, 1959) the form factors should be made dependent not only upon scalar products formed of the coordinate differences (2.9) but also upon a timelike four-vector. A possible candidate could be the four-momentum $\mathbf{P}_\mu$ (Wataghin 1959, Rayski 1968), because in terms of $\mathbf{P}_\mu$ one might define a center of mass system of coordinates, and introduce form factors being scalars assuming a desirable form (vanishing except for a microscopically extended domain) in the center-of-mass system. Unfortunately, this brings about serious complications: It means a dependence of the Lagrangian density upon the Hamiltonian, or upon the state vector since the latter determines which system of coordinates is the baricentric system. Such a theory does not seem to be possible or, at least, must be extremely complicated and rather artificial.

In what follows we shall discuss another possibility of securing macrocausality within the framework of manifestly nonlocal theories by taking advantage of the distinguished role played by the hypersurfaces of actual measurements σ_1 and σ_2. Namely, it is possible to relate with these hypersurfaces a field of timelike four-vectors. Let us consider a scalar field χ satisfying the Laplace equation:

$$\Box\chi = 0 \tag{2.12}$$

and fixed uniquely by the conditions

$$\chi_{[\sigma_1]} = C_1, \qquad \chi_{[\sigma_2]} = C_2 \tag{2.13}$$

where the notation $\chi_{[\sigma]}$ means the function χ taken at a point at the hypersurface σ. The equation

$$\chi = t \tag{2.14}$$

where t denotes a parameter, defines a timelike de Donder's coordinate

(harmonic coordinate of Fock). The values $t = C_1$ and $t = C_2$ denote the hypersurfaces σ_1 and σ_2, respectively.

The above-introduced time coordinate t is manifestly privileged with respect to the pair of hypersurfaces $\sigma_{1/2}$. Moreover, there appears a privileged timelike vector field

$$h_\mu = \partial_\mu \chi \tag{2.15}$$

Now, we come to the main point: If the field equations are necessarily dependent functionally on the hypersurfaces $\sigma_{1/2}$, why not make them dependent in another way as well, namely, by assuming that the form factor is also dependent on them, or rather upon the vector field h^μ

$$F = F(r', r'', r''', h) \tag{2.16}$$

whereby all the vectors $r'_\mu, r''_\mu, r'''_\mu$, and h_μ originate at the same point x^ν.

In particular, the form factor may be of the form

$$F(r', r'', r''', h) = G(r', r'', r''')\delta(r' \cdot h)\delta(r'' \cdot h)\delta(r''' \cdot h) \tag{2.17}$$

The appearance of Dirac delta functions in (2.17) means that such form factors smear out the interaction only in spacelike directions, orthogonally to the gradient of the field χ. For points situated on σ_1 or σ_2 they smear out the interaction only tangentially to these hypersurfaces. Such form factors may be called spatially nonlocal. The function G in (2.17) may be chosen so as to restrict this tangential smearing out only to microscopically small spacelike distances. In this way the manifestly nonlocal theory may be formulated in a way consistent with both the requirements of special relativity and macrocausality. On the other hand, similarly to the implicitly nonlocal theory, it violates microcausality and remains Lorentz covariant, only in a more restrictive sense than usual (conditioned Lorentz covariance).

3. *Quantization*

The differential equations of nonpolynomial, nonlocal or nonlinear theories with bounded interactions or the integrodifferential equations in the case of manifestly nonlocal theories may be replaced by pure integral equations with bounded kernels. They are inhomogeneous equations with the inhomogeneous terms representing free fields which determine the initial conditions either on σ_1 or on σ_2. Since the kernels are bounded, the solutions may be obtained by means of a perturbation calculus, or iteration, starting with the free fields satisfying the same initial conditions as the interacting fields. The perturbation series may be shown to be convergent if the kernels are bounded and the domain enclosed between σ_1 and σ_2 is finite.

Field quantization may be achieved in a most straightforward way by first quantizing the free fields in the MWPR described in Section 1 of this paper and then computing the interacting field operators by means of the perturbation calculus. A unitary operator U_{21} transforming the fields on σ_1 into those on σ_2 exists and may be computed as well by the perturbation calculus.

On the other hand, the problem of existence of the S operator, i.e., of the limit transitions with σ_1 to minus infinity and σ_2 to plus infinity is still problematic. But the existence of an S matrix is not really a necessary condition for a meaningful, physically well-founded scattering theory. This last always concerns, in reality, wave packets of finite extensions.

4. *Renormalization*

The theories with bounded interactions are free of the usual, ultraviolet divergences. Of course, the masses and coupling constants are to be renormalized, but the respective renormalization constants are finite in this case. The only infinite constant is the vacuum self-energy in the case of a field considered in an infinite domain. This infinity is, however, not very harmful, owing to the fact that one can always subtract from the Hamiltonian an arbitrary constant without affecting the dynamics of the physical system.

The renormalization procedure is rather simple in the case of regular theories: The dressed rest masses and coupling constants m, g are functions of the bare parameters appearing originally in the Lagrangian: bare masses m_0, bare coupling constants g_0, and possibly also multiplicative parameters α_0 attached to the bare field quantities themselves,

$$m = m(m_0, g_0, \alpha_0), \qquad g = g(m_0, g_0, \alpha_0), \qquad \alpha = \alpha(m_0, g_0, \alpha_0) \tag{4.1}$$

It is only necessary to solve these equations in the form

$$m_0 = m_0(m, g, \alpha), \qquad g_0 = g_0(m, g, \alpha), \qquad \alpha_0 = \alpha_0(m, g, \alpha) \tag{4.1'}$$

and fix the numerical values of m_0, g_0, and α_0 appearing originally in the Lagrangian by introducing into the right-hand sides of (4.1′) the actual, experimentally established values of m, g, and $\alpha = 1$. The theory is renormalizable if the equations (4.1) are soluble in the form (4.1′) and if the actual values of m, g, and $\alpha = 1$ belong to the domain of the functions (4.1′). No other criteria of renormalizability exist in the case of regular theories.

However, in the case of the (manifestly) nonlocal or nonlinear theories there appears an additional difficulty: the dressed parameters m, g computed within a theory that manifestly privileges the hypersurfaces of measurement (e.g., by making the form factor dependent on $\sigma_{1/2}$ or by defining the interaction in terms of a representation attached to one of them) might and almost certainly would depend functionally on these hypersurfaces or on the vector field h_μ given by (2.15). This would be inadmissible because physical masses and coupling constants are numbers independent of any measurement conditions. But the situation may be reversed by assuming the bare quantities to be some functionals of the hypersurfaces $\sigma_{1/2}$ (or of the vector field h_μ)

$$m_0 = m_0(m, g, \alpha, \sigma_1, \sigma_2), \qquad g_0 = g_0(m, g, \alpha, \sigma_1, \sigma_2),$$
$$\alpha_0 = \alpha_0(m, g, \alpha, \sigma_1, \sigma_2) \tag{4.2}$$

which is admissible because they are only auxiliary quantities deprived of any physical meaning. These functionals should be chosen so as to obtain for the

dressed, physical parameters m, g the values known from experiment, independent of the choice of σ_1 and σ_2. There is little doubt about the feasibility of this program.

5. *Outlook*

Hitherto, the traditional, local, polynomial, and renormalizable theories have been proved, at most (and taking the utmost pains in demonstrating it) to be free of the ultraviolet divergences (after regularization and renormalization) in every finite order of the perturbation calculus, but nobody was able to prove that the whole perturbative series converges—in other words, that the theory is meaningful. The situation is quite different in the cases of nonpolynomial, nonlinear, or nonlocal theories with bounded interactions. They constitute the first examples of *regular* quantum field theories for which the existence proofs are feasible. Under such circumstances the fact that a great majority of experts are exclusively interested in the traditional, local, polynomial theories shows only their extreme conservatism.

By introducing form factors of the type similar to (2.8) to electrodynamics, or restricting the electrodynamical interaction in a way similar to (2.3), (2.5), or (2.6), one would spoil gauge invariance. The breakdown of gauge invariance is, however, weak: global gauge invariance (gauge invariance of the first kind) still holds true, as well as the invariance of the second kind provided the gauge function $\theta(x)$ is sufficiently slowly variable so that on distances of the order of the constant l (determining the upper bound of the nonlinear interaction or the range of nonlocality) it may still be regarded as constant. Such weak violation of gauge invariance is physically acceptable.

Nevertheless, even a weak breakdown of gauge invariance may have unwanted effects, viz., production of a photon self-mass of a considerable magnitude. It is possible to avoid this difficulty by introducing into the Lagrangian another gauge-invariance-violating term, namely a bare photon rest-mass term so as to cancel the self-mass arising from the nonlocal interaction. This seems satisfactory in view of the fact that it is the dressed but not the bare rest mass of the photon that has to be exactly equal to zero. The requirement of having gauge covariance secured simultaneously in all orders of perturbation calculus may be a prejudice. The appearance of a nonvanishing "mechanical" mass may help to avoid the infrared difficulties appearing in the traditional electrodynamics.

A program for the future consists in a thorough investigation of all types of bounded interactions (nonpolynomial, nonlinear, nonlocal) in order to see which of them yield effectively nonvanishing and physically satisfactory results, in best agreement with the experimental evidence.

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